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ON ABSOLUTE WEIGHTED MEAN SUMMABILITY FACTOR OF AN INFINITE SERIES AND ITS APPLICATIONS

SANJAY TRIPATHI

Abstract

In the paper, we have proved a result on absolute summability factor method of an infinite series by using quasi $(\beta-\gamma)$- power increasing sequence, which generalizes some of the known results.

Keywords:

Infinite series;
Absolute Summability;
Summability Factors;
Almost increasing sequence;
Quasi β- Power increasing sequence.

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1. Introduction

A positive sequence (b_n) is said to be almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \le b_n \le Bc_n$. Every increasing sequence is almost increasing sequence but the converse need not be true as can be seen from the example, say $b_n = ne^{(-1)^n}$ (see[5]). A positive sequence (γ_n) is said to be a quasi $\beta-$ power increasing sequence if there is a constant $K = K(\beta,\gamma) \ge 1$ such that $Kn^{\beta}\gamma_n \ge m^{\beta}\gamma_m$ holds for all $n \ge m \ge 1$. It should be noted that every almost increasing sequence is quasi $\beta-$ power increasing sequence for any $\beta > 0$, but the converse need not be true as can be seen by example $\gamma_n = n^{-\beta}$ for $\beta > 0$. If $\beta = 0$, then (γ_n) is simply called a quasi increasing sequence.

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with (s_n) as the sequence of its partial sums. Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty, \text{ as } n \to \infty \quad (P_{-i} = p_{-i} = 0, i \geq 1).$$

The sequence – to – sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (t_n) of $(\overline{N}, p_n)$ transform of (s_n) generated by (p_n). The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $\left|\overline{N}_p, \phi_n; \delta\right|_k$, $k \geq 1$, $\delta \geq 0$ and $T = \delta k + k - 1$, if (see [6])

$$\sum_{n=1}^{\infty} \phi_n^T \left| t_n - t_{n-1} \right|^k < \infty,$$

where (ϕ_n) be any sequence of positive real constants.

Remarks : In particular case, we observed that

1. For $\delta = 0$, the summability $\left|\overline{N}_p, \phi_n; \delta\right|_k$ reduces to $\left|\overline{N}, p_n, \phi_n\right|_k$ summability due to W.T.Sulaiman [10]
2. For $\delta = 0$ and $\phi_n = \frac{P_n}{p_n}$, the summability $\left|\overline{N}_p, \phi_n; \delta\right|_k$ reduces to $\left|\overline{N}, p_n\right|_k$ summability due to H.Bor [1]
3. For $\phi_n = \frac{P_n}{p_n}$, the summability $\left|\overline{N}_p, \phi_n; \delta\right|_k$ reduces to $\left|\overline{N}, p_n; \delta\right|_k$ summability due to H.Bor [1].
4. If we put $\delta = 0$ and $\phi_n = n$, for all values of n, then $\left|\overline{N}_p, \phi_n; \delta\right|_k$ summability reduces to $\left|R, p_n\right|_k$ summability due to W.T.Sulaiman [9]
5. If $\phi_n = n$, for all values of n, the summability $\left|\overline{N}_p, \phi_n; \delta\right|_k$ reduces to $\left|R, p_n; \delta\right|_k$, summability due to W.T.Sulaiman [9]
6. If we take $\phi_n = \frac{P_n}{p_n}$ and $p_n = 1$ for all values of n, then $\left|\overline{N}_p, \phi_n; \delta\right|_k$ reduces to $\left|C, 1; \delta\right|_k$ summability which on putting $\delta = 0$ which becomes $\left|C, 1\right|_k$ due to T.M.Flett [8].

2. Main Result

The aim of this paper is to prove a result by considering $\left|\bar{N}_p,\phi_n;\delta\right|_k$ summability. In fact, we shall prove the following result

Theorem 1: Let (p_n) be a sequence of positive numbers such that

$$P_n = O(np_n) \text{ as } n \to \infty \quad . \tag{2.1}$$

If (X_n) be quasi $(\beta-\gamma)$- power increasing sequence for some $0<\beta<1$ and the sequences (λ_n) and (β_n) are such that

$$|\Delta\lambda_n| \le \beta_n \tag{2.2}$$

$$\beta_n \to 0 \quad as \quad n \to \infty \tag{2.3}$$

$$\sum_{n=1}^{\infty} nX_n|\Delta\beta_n| < \infty \tag{2.4}$$

$$|\lambda_n| X_n = O(1) \text{ as } n \to \infty, \tag{2.5}$$

$$\sum_{n=1}^{m} \phi_n^T \left(\frac{p_n}{P_n}\right)^k |s_n|^k = O(X_m) \qquad as \quad m \to \infty \tag{2.6}$$

and

$$\sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{T-k} \frac{1}{P_{n-1}} = O\left[\left(\frac{P_v}{p_v}\right)^{\tau k} \frac{1}{P_v}\right]. \tag{2.7}$$

where (ϕ_n) be a sequence of positive real constants such that $\left(\frac{\phi_n p_n}{P_n}\right)$ is non-increasing sequence ,

then the series $\sum_{n=0}^{\infty} a_n\lambda_n$ is summable $\left|\bar{N}_p,\phi_n;\delta\right|_k$, $k \ge 1$ and $0 \le \tau < \frac{1}{k}$.

3. Lemma:

We need the following lemma for the proof our result.

Lemma 1 [11, lemma 2.2] : Let (X_n) quasi $(\beta-\gamma)$ - power increasing sequence, $0<\beta<1$ and $\gamma \ge 0$, then the condition (2.3) and (2.4) implies

$$n\beta_n X_n < \infty \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty \tag{3.2}$$

4. Proof of the Theorem 1:

Let (t_n) be the sequence of $(\overline{N}, p_n)$ means of the series $\sum_{n=0}^{\infty} a_n \lambda_n$, then, by definition, we have

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{i=0}^{v} a_i \lambda_i = \frac{1}{P_n}\sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v$$

Then, for $n \geq 1$ and by using simple calculation, we get

$$t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v \tag{4.1}$$

Using Able's transformation to the right hand side of (4.1), we get

$$t_n - t_{n-1} = \frac{p_n s_n \lambda_n}{P_n} - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v$$

$$= t_{n,1} + t_{n,2} + t_{n,3} \text{, say}.$$

Since

$$\left| t_{n,1} + t_{n,2} + t_{n,3} \right|^k \leq 3^k \left(\left|t_{n,1}\right|^k + \left|t_{n,2}\right|^k + \left|t_{n,3}\right|^k \right).$$

Thus, in order to complete the proof of the Theorem 1, it is sufficient to show that

$$\sum_{n=1}^{\infty} \phi_n^T \left|t_{n,z}\right|^k < \infty \text{ , for } z = 1,2,3.$$

We have,

$$\sum_{n=1}^{m} \phi_n^T \left| t_{n,1} \right|^k$$

$$= \sum_{n=1}^{m} \phi_n^T \left| \frac{p_n s_n \lambda_n}{P_n} \right|^k$$

$$= O(1) \sum_{n=1}^{m} \phi_n^T \left(\frac{p_n}{P_n}\right)^k |s_n|^k (|\lambda_n|)^{k-1} |\lambda_n|$$

$$= O(1) \sum_{n=1}^{m} \phi_n^T \left(\frac{p_n}{P_n}\right)^k |s_n|^k |\lambda_n| \text{, by (2.5)}$$

$$= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} \phi_v^T \left(\frac{p_v}{P_v}\right)^k |s_v|^k + O(1)\ |\lambda_m| \sum_{n=1}^{m} \phi_n^T \left(\frac{p_n}{P_n}\right)^k |s_n|^k$$

$$= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n + O(1)\ |\lambda_m|\ X_m, \quad \text{by (2.6)}$$

$$= O(1)\sum_{n=1}^{m-1}\beta_n X_n + O(1)\ |\lambda_m|\ X_m, \quad \text{by (2.2)}$$

$= O(1)$ as $m \to \infty$, by ((3.2) and (2.5)).

Again,

$$\sum_{n=2}^{m+1}\phi_n^T\left|t_{n,2}\right|^k$$

$$= \sum_{n=2}^{m+1}\phi_n^T\left|\frac{-p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_v s_v\lambda_v\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}\phi_n^T\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left\{\sum_{v=1}^{n-1}p_v|s_v||\lambda_v|\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^T\left(\frac{P_n}{p_n}\right)^{T-k}\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}p_v|s_v|^k|\lambda_v|\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m}p_v|s_v|^k|\lambda_v|\sum_{n=v+1}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^T\left(\frac{P_n}{p_n}\right)^{T-k}\frac{1}{P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{\phi_v p_v}{P_v}\right)^T p_v|s_v|^k|\lambda_v|\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{T-k}\frac{1}{P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\phi_v^T\left(\frac{p_v}{P_v}\right)^T\left(\frac{P_v}{p_v}\right)^{T-k}|s_v|^k|\lambda_v|, \quad \text{by (2.7)}$$

$$= O(1)\sum_{v=1}^{m}\phi_v^T\left(\frac{p_v}{P_v}\right)^k|s_v|^k|\lambda_v|$$

$= O(1)$ *as* $m \to \infty$, Proceeding as in case $\left|t_{n,1}\right|$

Finally we have,

$$\sum_{n=2}^{m+1}\phi_n^T\left|t_{n,3}\right|^k$$

$$= \sum_{n=2}^{m+1}\phi_n^T\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v s_v\Delta\lambda_v\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}\phi_n^T\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left\{\sum_{v=1}^{n-1}p_v v|s_v||\Delta\lambda_v|\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^T\left(\frac{P_n}{p_n}\right)^{T-k}\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}p_v|s_v|^k(v\beta_v)^k\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^T\left(\frac{P_n}{p_n}\right)^{T-k}\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}p_v|s_v|^k(v\beta_v)^k\right\}$$

$$= O(1)\sum_{v=1}^{m}p_v|s_v|^k(v\beta_v)\sum_{n=v+1}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^T\left(\frac{P_n}{p_n}\right)^{T-k}\frac{1}{P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\phi_v^T\left(\frac{p_v}{P_v}\right)^k|s_v|^k(v\beta_v) \quad \text{by (2.7)}$$

$$= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|\sum_{i=1}^{v}\phi_i^T\left(\frac{p_i}{P_i}\right)^k|s_i|^k + O(1)\; m\beta_m\sum_{i=1}^{m}\phi_i^T\left(\frac{p_i}{P_i}\right)^k|s_i|^k$$

$$= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v + O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_{v+1}| + O(1)\; m\beta_m X_m,$$

$$= O(1)\sum_{v=1}^{m-1}v|\beta_v|X_v + O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_v + O(1)\; m\beta_m X_m$$

$$= O(1) \quad as \quad m\to\infty, \text{ by (2.4), (3.2) and (3.1).}$$

Thus we have shown that

$$\sum_{n=1}^{\infty}\phi_n^T|t_{n,z}|^k<\infty \text{ , for } z=1,2,3.$$

which completes the proof of the Theorem 1.

5. Applications:

If we consider the special cases of our Theorem 1, then following results are the consequences of our Theorem 1, which we have put in the form of corollaries as follows:

Corollary 1 : It must be noted that, every almost increasing sequence is quasi $(\beta-\gamma)$- power increasing sequence for $\gamma=0$.Thus, Theorem 1 generalizes our result [7].

Corollary 2: If $\delta=0$ and $\phi_n=\frac{P_n}{p_n}$, then our results (Theorem 1) reduces for $|\bar{N},p_n|_k$ summability ,which extend the result of [2].

Corollary 3: If $\delta=0$, then our results (Theorem 1) reduces for $|\bar{N},p_n,\phi_n|_k$ summability.

Corollary 4 : If $\phi_n = \frac{P_n}{p_n}$, then our results (Theorem 1) reduces for $\left|\overline{N}, p_n; \delta\right|_k$ summability ,which extend the result of[3].

Corollary 5 : If $\delta = 0$ and $\phi_n = n$ for all values of n, then our results (Theorem 1) reduces for $\left|R, p_n\right|_k$ summability.

Corollary 6 : If $\phi_n = n$ for all values of n, then our results (Theorem 1) reduces for $\left|R, p_n; \delta\right|_k$ summability.

Corollary 7 : If $\phi_n = \frac{P_n}{p_n}$ and $p_n = 1$ for all values of n, then our results (Theorem 1) reduces for $\left|C,1;\delta\right|_k$ summability.

Corollary 8 : If $\phi_n = \frac{P_n}{p_n}$ and $\delta = 0$ and $p_n = 1$ for all values of n, then our results (Theorem 1) reduces for $\left|C,1\right|_k$ summability (see[4]).

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